Lights Out

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The Game

Lights Out is a game that consists of a 5×5 grid of buttons that light up. The light of each button can be turned on or off by pressing it. If a button is pressed, though, it changes the status of the four buttons adjacent to it as well. If the button is pressed and a neighboring button is off, that neighbor will be turned on, and if the neighbor is on it will be turned off. The goal of the game is to turn off all of the lights in the grid.

Note that because buttons can only be on or off and the pattern of the change in states of adjacent buttons is constant, if a button is pushed twice (or any even number of times, for that matter), the pattern of lit and unlit buttons remains the same as it was before the button was pushed. It is as if the button was never pushed in the first place. Since each button has only two states, all of our arithmetic will be done in the scalar field $\mathbb{Z}_2$.

Also note that because the status of a button depends solely on what state it was in to begin with and how many times it and its neighbors have been pressed, it does not matter in which order buttons are pressed.

Pattern and Change Vectors

We start with the grid set up in a certain pattern of buttons on and off. Let's turn this grid into a vector of 25 elements. Each element represents one of the buttons. Let's represent a button that's off as 0 and a lit button as 1. Let's label the buttons as the following:

$g_{1,1}$	$g_{1,2}$	$g_{1,3}$	$g_{1,4}$	$g_{1,5}$
$g_{2,1}$	$g_{2,2}$	$g_{2,3}$	$g_{2,4}$	$g_{2,5}$
$g_{3,1}$	$g_{3,2}$	$g_{3,3}$	$g_{3,4}$	$g_{3,5}$
$g_{4,1}$	$g_{4,2}$	$g_{4,3}$	$g_{4,4}$	$g_{4,5}$
$g_{5,1}$	$g_{5,2}$	$g_{5,3}$	$g_{5,4}$	$g_{5,5}$

.

So we have the representative vector

$$\mathbf{g} = \begin{bmatrix} g_{1,1} \\ g_{1,2} \\ \vdots \\ g_{1,5} \\ g_{2,1} \\ \vdots \\ g_{5,5} \end{bmatrix}.$$

Now we can represent the change that occurs when a button is pressed as a vector. Let's have a button that does not change represented by 0 and one that changes status represented by 1. So our vector of change is

$$\mathbf{c} = \begin{bmatrix} c_{1,1} \\ c_{1,2} \\ \vdots \\ c_{1,5} \\ c_{2,1} \\ \vdots \\ c_{5,5} \end{bmatrix}.$$

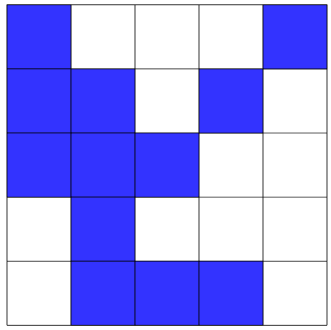
The result of the button press is then the addition of the two vectors in the scalar field $\mathbb{Z}_2$,

$$\mathbf{n} = \mathbf{g} + \mathbf{c} = \begin{bmatrix} g_{1,1} + c_{1,1} \\ g_{1,2} + c_{1,2} \\ \vdots \\ g_{1,5} + c_{1,5} \\ g_{2,1} + c_{2,1} \\ \vdots \\ g_{5,5} + c_{5,5} \end{bmatrix},$$

and this is now our new grid pattern.

Example 1: A grid and its pattern and change vectors

Say we begin with a grid that looks like



where a shaded box represents a light that is on and an unshaded box represents a light that is off.

So we have

$$\mathbf{g} = \begin{bmatrix}1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0\end{bmatrix}^t.$$

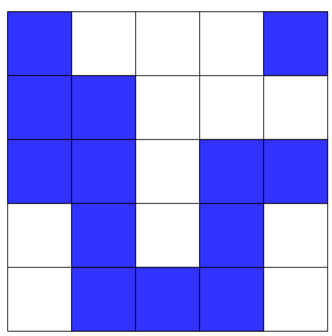
Let's say we push button $g_{3,4}$ on this grid. Then the change vector is

$$\mathbf{c} = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^t.$$

When we add $\mathbf{g}$ and $\mathbf{c}$ in mod 2 we get

$$\mathbf{n} = \mathbf{g} + \mathbf{c} = \begin{bmatrix}1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0\end{bmatrix}^t,$$

which gives us the new grid pattern of



.

Because addition is commutative, we can add vectors in any order and get the same result. Therefore we can press buttons in any order and end up with the same grid pattern.

The Strategy Vector

Let our button-pressing strategy be represented by the vector $\mathbf{s}$ where $s_{i,j}$ is equal to 1 if button (i, j) is the button we push and 0 if we leave it alone. In this paper, the strategy vector may sometimes also be called the solution vector.

Example 2: A strategy vector

In Example 1 above, where we press button $g_{3,4}$, the strategy is represented by

$$\mathbf{s} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^t.$$

A Matrix of Change Vectors

Now, in order to simplify our manipulation of the buttons of the grid we will build the matrix whose columns are all the possible change vectors. In order to create this matrix easily and to be sure we find every change vector, we'll name each button in this way:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

.

We are going to build the 25×25 matrix A which will represent what each button does to the rest of the lights on the grid. Let $a_{i,j}$ be 1 if light i is changed by button j, and 0 otherwise. Here is A:

$$A = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}.$$

In order to simplify this matrix, we can see that

$$A = \begin{bmatrix}
B & I & \mathcal{O} & \mathcal{O} & \mathcal{O} \\
I & B & I & \mathcal{O} & \mathcal{O} \\
\mathcal{O} & I & B & I & \mathcal{O} \\
\mathcal{O} & \mathcal{O} & I & B & I \\
\mathcal{O} & \mathcal{O} & \mathcal{O} & I & B
\end{bmatrix},$$

where I is the 5×5 identity matrix, $\mathcal{O}$ is the 5×5 zero matrix, and

$$B = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}.$$

Since B is a symmetric matrix, A is symmetric as well.

Finding a Strategy that Solves a Grid Pattern

Say we begin with grid pattern $\mathbf{g}$. Then $\mathbf{g}$ is solvable if there exists a strategy $\mathbf{s}$ that turns off all the lights in $\mathbf{g}$ (that turns $\mathbf{g}$ into the zero vector). Also note that if a set of buttons is pushed to create a grid pattern, then starting with that grid pattern and pressing the same set of buttons will turn the lights off.

To find a strategy to turn off all the lights in $\mathbf{g}$, we need to solve $\mathbf{g} = A\mathbf{s}$. So grid pattern $\mathbf{g}$ is only solvable if and only if it belongs to the column space of matrix A, $\mathcal{C}(A)$.

In order to find $\mathcal{C}(A)$, we can put matrix A into reduced-row echelon form using Gauss-Jordan elimination in mod 2. When we augment A with the 25×25 identity matrix and row reduce, we get $RA = E$ where R is the elementary matrix of size 25 that produces the row operations that row-reduce matrix A, and E is the matrix in reduced-row echelon form. Here are R and E:

$$R = \begin{bmatrix}
0&1&1&1&0&0&0&1&0&1&0&0&0&1&1&0&0&0&0&1&0&0&0&0&0\\
1&1&0&1&1&0&1&0&0&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0\\
1&0&1&1&1&1&0&1&1&0&0&0&1&1&0&1&1&1&1&1&0&1&0&0&0\\
1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0\\
0&1&1&0&1&1&0&0&0&0&1&0&1&0&1&0&0&1&0&1&1&1&0&0&0\\
0&0&1&0&1&0&1&1&0&1&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0\\
0&1&0&1&0&1&1&0&1&1&0&0&0&1&0&1&1&1&0&0&0&1&0&0&0\\
1&0&1&0&0&1&0&1&1&0&0&0&0&0&1&1&0&1&0&1&1&0&1&0&0\\
0&0&1&0&0&0&1&1&1&0&1&0&0&1&1&1&0&0&1&0&0&1&1&0&0\\
1&0&0&0&0&1&1&0&0&0&1&0&1&0&1&0&1&1&0&1&0&0&1&0&0\\
0&0&0&0&1&0&0&0&1&1&0&0&1&0&1&1&1&1&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0\\
0&1&1&0&1&1&0&0&0&1&1&0&1&1&0&0&0&1&0&0&1&1&0&0&0\\
1&1&1&0&0&0&1&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0\\
1&1&0&0&1&0&0&1&1&1&1&0&0&1&1&1&1&0&1&0&1&0&0&0&0\\
0&0&1&0&0&0&1&1&1&0&1&0&0&0&1&1&0&1&0&1&1&0&1&0&0\\
0&0&1&1&0&0&1&0&0&1&1&1&0&0&1&0&1&1&1&0&0&0&1&0&0\\
0&0&1&0&1&0&1&1&0&1&1&0&1&0&0&1&1&0&1&1&1&0&0&0&0\\
0&1&1&0&0&1&0&0&1&0&1&0&0&1&1&0&1&1&1&0&0&0&1&0&0\\
1&0&1&0&1&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&1&0&1&0&0\\
0&0&0&1&1&0&0&1&0&0&0&1&1&0&1&1&0&1&0&1&1&1&0&0&0\\
0&0&1&1&1&0&1&0&1&0&1&1&1&0&0&0&0&0&0&0&1&1&1&0&0\\
0&0&0&1&0&0&0&1&1&1&0&1&0&0&0&1&1&0&1&1&0&1&0&0&0\\
0&1&1&1&0&1&0&1&0&1&1&1&0&1&1&1&0&1&0&1&0&1&1&1&0\\
1&0&1&0&1&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&1&0&1&0&1
\end{bmatrix}$$

$$E = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

We can now see that $\mathcal{C}(A)$ is equal to the span of the column vectors that form the matrix E. Note that in matrix E there are 5 rows that contain only a single 1, so there are only 5 lights that can be turned on or off individually. Also, if we add the rows of E together, we will get 23 ones. The last two columns of E also add to ones. This shows that it is possible to switch on all the lights. To do so, we find the strategy given by the sum of all the rows of matrix R.

Gauss-Jordan elimination gives us E, which is in reduced-row echelon form. We find that the matrix E has rank 23 with 2 free variables, $s_{5,4}$ and $s_{5,5}$. The last two columns of E are

$$\begin{bmatrix}0&1&1&1&0&1&0&1&0&1&1&1&0&1&1&1&0&1&0&1&0&1&1&0&0\end{bmatrix}^t$$

and

$$\begin{bmatrix}1&0&1&0&1&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&1&0&1&0&0\end{bmatrix}^t.$$

Note that A is a symmetric matrix, so the column space of A is equal to the row space of A, $\mathcal{C}(A) = \mathcal{R}(A)$. But $\mathcal{R}(A)$ is the orthogonal complement of the null space of A, $\mathcal{N}(A)$, which is equal to $\mathcal{N}(E)$. So to find $\mathcal{C}(A)$, we simply need to find a basis for $\mathcal{N}(E)$. We can find an orthogonal basis for $\mathcal{N}(E)$ by looking at the last 2 columns of E. The vectors in our orthogonal basis for $\mathcal{N}(E)$ are as follows:

$$\mathbf{m}_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}^t$$

and

$$\mathbf{m}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^t.$$

Since $\mathbf{m}_1$ and $\mathbf{m}_2$ are vectors in the basis of $\mathcal{N}(E)$, when we press the buttons indicated in these vectors or the sum of these vectors, we end up with the same grid pattern that we began with.

Solvable Grid Patterns

A grid pattern $\mathbf{g}$ is solvable only if $\mathbf{g}$ is orthogonal to the two vectors $\mathbf{m}_1$ and $\mathbf{m}_2$. To find out if a pattern is solvable, find the inner product of that pattern with $\mathbf{m}_1$ and $\mathbf{m}_2$. We can see that there are 2^{25} possible grid patterns since there are 25 lights and 2 possible states for each light. We also know the nullity of E, $n(E) = \dim(\mathcal{N}(E)) = 2$, and since we are working in the $\mathbb{Z}_2$ scalar field, we can see that out of the 2^{25} possible patterns, one-fourth of them are solvable.

Example 3: A solvable pattern vector

Say we have a pattern vector

$$\mathbf{f} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}^t.$$

In mod 2 $\langle \mathbf{f}, \mathbf{m}_1 \rangle = 0$ and $\langle \mathbf{f}, \mathbf{m}_2 \rangle = 0$, which shows that $\mathbf{f}$ is orthogonal to both $\mathbf{m}_1$ and $\mathbf{m}_2$, and therefore it is a solvable pattern.

Example 4: An unsolvable pattern vector

Say we have a pattern vector

$$\mathbf{h} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^t.$$

In mod 2 $\langle \mathbf{h}, \mathbf{m}_1 \rangle = 0$ and $\langle \mathbf{h}, \mathbf{m}_2 \rangle = 1$, which shows that $\mathbf{h}$ is orthogonal to $\mathbf{m}_1$, but it is not orthogonal to $\mathbf{m}_2$, so it is not a solvable pattern, and there is no solution vector that will take $\mathbf{h}$ to $\mathbf{0}$.

If $\mathbf{g}$ is a solvable pattern with the winning strategy $\mathbf{s}$, then $\mathbf{s} + \mathbf{m}_1$, $\mathbf{s} + \mathbf{m}_2$, and $\mathbf{s} + \mathbf{m}_1 + \mathbf{m}_2$ are also winning strategies, since $\mathbf{s} \in \mathcal{N}(E)$ where $\mathcal{N}(E) = \langle \{\mathbf{m}_1, \mathbf{m}_2\} \rangle$.

Multiple Strategies

Now suppose that $\mathbf{g}$ is a solvable pattern. We want to find one of the four strategies $\mathbf{s}$ for which $A\mathbf{s} = \mathbf{g}$. Since we only need one solution, we will let $s_{5,4}$ and $s_{5,5}$ equal zero. In this case $\mathbf{s} = E\mathbf{s}$. so $\mathbf{s} = E\mathbf{s} = RA\mathbf{s} = R\mathbf{g}$. So we have a strategy given by $\mathbf{s} = R\mathbf{g}$. Suppose that $\mathbf{g}$ is a solvable pattern. Then the four winning strategies for $\mathbf{g}$ are $R\mathbf{g}$, $R\mathbf{g} + \mathbf{m}_1$, $R\mathbf{g} + \mathbf{m}_2$, and $R\mathbf{g} + \mathbf{m}_1 + \mathbf{m}_2$.

Example 5: Find a winning strategy

Let's take our previous solvable pattern vector from Example 3. To find a winning strategy, we compute

$$\mathbf{s} = R\mathbf{f} = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0\end{bmatrix}^t.$$

So if we press buttons $f_{2,2}$, $f_{2,4}$, $f_{3,3}$, $f_{4,2}$, $f_{4,4}$, $f_{5,1}$, and $f_{5,3}$ we can turn off all the lights in the grid. If we want to minimize the number of button presses, we can calculate $\mathbf{s}+\mathbf{m}_1$, $\mathbf{s}+\mathbf{m}_2$, and $\mathbf{s}+\mathbf{m}_1+\mathbf{m}_2$ as well, and we can find the strategy with the minimum presses.

Now we know how to solve any (solvable) grid pattern!

Example 6: Finding a solution to our first puzzle

Let's find the solution to the pattern vector $\mathbf{g}$ that we have been working with throughout our discussion. We have our vector

$$\mathbf{g} = \begin{bmatrix}1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0\end{bmatrix}^t.$$

We find

$$\mathbf{s} = R\mathbf{g} = \begin{bmatrix}0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^t,$$

which has 14 button presses. We also find that

$$\mathbf{s}+\mathbf{m}_1 = \begin{bmatrix}0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0\end{bmatrix}^t,$$

which has 14 button presses. We calculate

$$\mathbf{s}+\mathbf{m}_2 = \begin{bmatrix}1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1\end{bmatrix}^t,$$

and

$$\mathbf{s}+\mathbf{m}_1+\mathbf{m}_2 = \begin{bmatrix}1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1\end{bmatrix}^t,$$

which each have 16 button presses, so we see that $\mathbf{s}$ and $\mathbf{s}+\mathbf{m}_1$ both solve the puzzle in the minimum number of presses.

Checking the Solution

In order to check that these solutions truly work, we multiply each of our four solution vectors with matrix A, and we should end up with our original pattern vector.

$$A\mathbf{s} = A(\mathbf{s}+\mathbf{m}_1) = A(\mathbf{s}+\mathbf{m}_2) = A(\mathbf{s}+\mathbf{m}_1+\mathbf{m}_2) = \mathbf{g}.$$

Since we found $\mathbf{s}$ for the equation $A\mathbf{s} = \mathbf{g}$, we should find that $A\mathbf{s}+\mathbf{g} = \mathbf{0}$, because we are working in mod 2, and this gives us our new pattern vector with all the lights turned off.

A Simple Algorithm

Now that we know how to calculate the solution strategy, we'll use a much simpler, algorithmic way to solve the game. Say we only compute the strategy for the first row of our grid (the first five entries in column R$\mathbf{g}$). We will take the first five entries in column $\mathbf{g}$ and multiply them with the 5×5 matrix composed of the first five entries in the first five rows and first five columns of matrix R, which we will call R'. We can also find R' by creating the 5×5 matrix whose columns are the change vectors for the first row of buttons, A'. We can then row-reduce A', and R' will be the elementary matrix that produces the row operations that put A' into reduced-row echelon form. We will call our shortened pattern vector $\mathbf{g}'$.

Example 7: Finding the strategy vector for the first row

$$\mathbf{g}' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } A' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

$$R' = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \text{ and } R'A' = E' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We find

$$\mathbf{s}' = R'\mathbf{g}' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Now we perform these moves on the first row. We have already established that each button only needs to be pressed once, so we know that no more moves in the first row are necessary. Now we check to see if any lights are still on in the first row. The only way to turn these off without pushing buttons in the first row is to push the buttons in the second row right below those that are still on. Then we do the same for the next rows, and we end up with all the lights turned off.

Alternate Versions of *Lights Out*

Lit buttons only

There is a version of *Lights Out* where, when solving the puzzle, only lit buttons may be pressed. It is possible to turn off all of the lights on a solvable grid pattern by doing this, though a few more buttons may have to be pressed. Once the strategy vector is found, we can press the buttons indicated by the strategy vector that are already lit on our game board. We keep doing this, pressing each button in our strategy vector only once, until the rest of the buttons in the strategy are unlit. If a lit button, x is adjacent to one that is unlit but is in our strategy, w, we can press the combination x, w, x. This has the same effect as pressing w alone, but it overcomes the hindrance of x being unlit. By doing this with every unlit solution button and one of its lit neighbors, we can eventually come to a pattern where the remaining buttons in our solution are all lit.

3-State Game

There are some versions of *Lights Out* that allow us to play the game in which each button has three different states - off and two different colors. When we press a button that is unlit, it turns on and becomes one color, and when we press it again it becomes another color. When we press it a third time, it turns off again. Similarly, the pressing of one button advances each neighboring button to its next state in the sequence. In this game, we are working in mod 3 with the scalar field of $\mathbb{Z}_3$.

Lights Out on an n×n Grid

When we find the solution to an $n \times n$ puzzle, we will be creating $n \times 1$ pattern vectors, change vectors, and solution vectors, and our matrix formed by all possible change vectors (matrix A in this case) will be a square matrix of size n^2. For each grid size, we will have a different null space of A, creating a different basis and a different number of possible solutions. In some cases, such as the case of a $6 \times 6, 7 \times 7$, or 8×8 grid, $\mathcal{N}(A) = 0$ because A is nonsingular. In these cases, since there are no dependent variables and since the basis of A is simply the span of the set of n standard unit column vectors, every possible grid pattern is solvable, and each solution, as long as no button is pressed more than once, is unique.

Lights Out on a Torus

Lights Out can be played in such a way that the buttons of the top row are adjacent to those of the bottom row, and the buttons of the left column are neighbors with those in the right column. In order to solve this puzzle, we change matrix A to reflect this property, and we find our strategy vector in the same manner as before.

Lights Out on an n×n Torus

This game is a combination of *Lights Out* on an $n \times n$ grid and *Lights Out* on a torus. In this case, the change vectors reflect the torus property of the grid, and the $n^2 \times n^2$ matrix A is formed with these new change vectors. The number of possible strategy vectors is again determined by $\mathcal{N}(A)$.

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